

Estimates of sections of determinant line bundles on Moduli spaces of pure sheaves on algebraic surfaces

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Abstract: Let X be any smooth simply connected projective surface. We consider some moduli space of pure sheaves of dimension one on X , i.e. $M_X^H(u)$ with $u = (0, L, \chi(u) = 0)$ and L an effective line bundle on X , together with a series of determinant line bundles associated to $r[\mathcal{O}_X] - n[\mathcal{O}_{pt}]$ in Grothendieck group of X . Let g_L denote the arithmetic genus of curves in the linear system $|L|$. For $g_L \leq 2$, we give a upper bound of the dimensions of sections of these line bundles by restricting them to a generic projective line in $|L|$. Our result gives, together with Göttsche's computation, a first step of a check for the strange duality for some cases for X a rational surface.

1 Introduction.

let X be a smooth complex projective surface with H an ample divisor, and u and c_n^r two elements in the Grothendieck group $\mathbf{K}(X)$ of X which are specified as $u = (0, L, \chi(u) = 0)$ for L an effective line bundle on X , and $c_n^r = r[\mathcal{O}_X] - n[\mathcal{O}_{pt}]$ where \mathcal{O}_{pt} is the skyscraper sheaf supported at a point in X . Denote $M_X^H(u)$ (resp. $M_X^H(c_n^r)$) the moduli space of semistable sheaves with respect to H on X of class u (resp. c_n^r). There is a so-called determinant line bundle $\lambda_{c_n^r}$ (resp. λ_u) on $M_X^H(u)$ (resp. $M_X^H(c_n^r)$) associated to c_n^r (resp. u) (See [5] Chapter 8 for more details). It is conjectured by Strange Duality that there is a natural isomorphism between the following two spaces (see [2] for more details)

$$D : H^0(M_X^H(u), \lambda_{c_n^r})^\vee \rightarrow H^0(M_X^H(c_n^r), \lambda_u). \quad (1.1)$$

We are concerned on the numerical version of the conjecture. In other words, we would like to check the following equality

$$h^0(M_X^H(u), \lambda_{c_n^r}) = h^0(M_X^H(c_n^r), \lambda_u). \quad (1.2)$$

In [8] for $X = \mathbb{P}^2$ or $\mathbb{P}(\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(-e))$ with $e = 0, 1$ and $L = 2G + aF$ with $2e \leq a \leq e + 3$ where F is the fiber class and G is the section such that $G.G = -e$, we have computed the generating function

$$Z^r(t) = \sum_{n \geq 0} h^0(M_X^H(u), \lambda_{c_n^r}) t^n, \quad (1.3)$$

for all $r \geq 1$. Moreover when $r = 2$, the result matches Göttsche's computation on the rank 2 sheaves side and gives a numerical check of Strange Duality for these cases (See [8] Corollary 4.4.2 and Corollary 4.5.3).

In this paper we consider more general cases. We ask X to be any smooth simply connected projective surface over the complex number \mathbb{C} . Let K be the canonical divisor of X . Let $|L|$ be the linear system associated to the line bundle L and l the dimension of $|L|$. Let g_L be the arithmetic genus of curves in $|L|$. For any two line bundles L and L' , we denote $L.L'$ to be the intersection number of their divisors; and moreover we write $L' \leq L$ if $L \otimes L'^{-1}$ is an effective line bundle, i.e. $h^0(L \otimes L'^{-1}) \neq 0$; and write $L' < L$ if $L' \leq L$ and $L' \neq L$. We state two assumptions on L as follows which are all the assumptions we need

$$(\mathbf{A}'_1) \quad L.K < 0;$$

(\mathbf{A}'_2) For any $0 < L', L'' < L$ with $L' + L'' = L$, we have $l' + l'' \leq l - 2$ where $l' = \dim |L'|$ and $l'' = \dim |L''|$.

Since we deal with more general cases, the techniques we used in [8] to obtain the normality and irreducibility of the Moduli space $M_X^H(u)$ and the dualizing sheaf on $M_X^H(u)$ don't work any more. We thus lose many good properties of the moduli spaces, but anyway we still have some results providing an estimate for the dimension of sections of $\lambda_{c_n^r}$ on $M_X^H(u)$. We have obtained in this paper the following three theorems:

Theorem 1.1. *Let X be simply connected and let L satisfy (\mathbf{A}'_1) and (\mathbf{A}'_2) . Then we have for all $n \geq 0$*

$$h^0(M(c_n^1), \lambda_u) \geq h^0(M(u), \lambda_{c_n^1}).$$

Moreover for any fixed r , once the strict inequality holds for $n = n_0$, it holds for all $n \geq n_0$.

Denote

$$Y_{g_L=1}^r(t) = \sum_{n \geq 0} y_{n,g_L=1}^r t^n = \frac{1 + t^2 + t^3 + \dots + t^r}{(1-t)^2};$$

and let $y_{n,g_L=1}^r = 0$ for all $n < 0$. Then we have

Theorem 1.2. *Let X be a smooth simply connected projective surface and L satisfy (\mathbf{A}'_1) and (\mathbf{A}'_2) with $g_L = 1$. Then we have for all $n \in \mathbb{Z}$ and $r \geq 1$,*

$$y_{n,g_L=1}^r \geq h^0(M(u), \lambda_{c_n^r}).$$

Moreover for any fixed r , once the strict inequality holds for $n = n_0$, it holds for all $n \geq n_0$.

Let $Y_{g_L=2}^1 = \sum_n y_{n,g_L=2}^1 t^n = \frac{1}{(1-t)^2}$ and for $r \geq 2$

$$Y_{g_L=2}^r(t) = \sum_n y_{n,g_L=2}^r t^n = \frac{1 + 3t^2 + \sum_{i=3}^r ((i+1)t^i + (i-2)t^{i+1})}{(1-t)^{l+1}}.$$

Let $y_{n,g_L=2}^r = 0$ for all $n < 0$. Then we have

Theorem 1.3. *Let X be a smooth simply connected projective surface and L satisfy (\mathbf{A}'_1) and (\mathbf{A}'_2) with $g_L = 2$ and $\dim |L| \geq 3$. Then we have for all $n \in \mathbb{Z}$ and $r \geq 1$,*

$$y_{n,g_L=2}^r \geq h^0(M(u), \lambda_{c_n^r}).$$

Moreover for any fixed r , once the strict inequality holds for $n = n_0$, it holds for all $n \geq n_0$.

Remark 1.4. *Fix $r = 2$. Göttsche's results for rational ruled surfaces together with his blow-up formulas give many examples for X a rational surface, in which L satisfies (\mathbf{A}'_1) and (\mathbf{A}'_2) with $g_L = 1$ or $g_L = 2$ and $l \geq 3$, and also the following equalities hold under some suitable polarization (a change of the polarization may give a difference of a polynomial)*

$$\sum_{n \geq 0} \chi(M(c_n^2), \lambda_L) t^n = \frac{1 + t^2}{(1+t)^{l+1}} = Y_{g_L=1}^2(t), \text{ if } g_L = 1;$$

$$\sum_{n \geq 0} \chi(M(c_n^2), \lambda_L) t^n = \frac{1 + 3t^2}{(1+t)^{l+1}} = Y_{g_L=2}^2(t), \text{ if } g_L = 2.$$

Hence we have for these cases under a suitable polarization for all $n \geq 0$

$$\chi(M(c_n^2), \lambda_u) \geq h^0(M(u), \lambda_{c_n^2}).$$

In particular (under any polarization) for $n \gg 0$, we have

$$\chi(M(c_n^2), \lambda_u) = h^0(M(c_n^2), \lambda_u) \geq h^0(M(u), \lambda_{c_n^2}).$$

The main idea to prove these three theorems is to restrict Θ^r to intersections of pull back of hyperplanes in $|L|$ until finally we reach a generic projective line T in $|L|$. We then compute the splitting type of $\pi_*(\Theta^r|_{\pi^{-1}(T)})$ on T . We prove Theorem 1.1 in Section 4, Theorem 1.2 in Section 5. The proof of Theorem 1.3 is the most complicated one among the three and is done in Section 6. Also in Section 6 we obtain a corollary (Corollary 6.16) in the theory of compactified Jacobian of integral curves with planar singularities.

2 Notations.

Let u_χ be an element in $\mathbf{K}(X)$ given by $u_\chi = (0, L, \chi(u_\chi) = \chi)$, and M_χ the moduli space of semistable sheaves (w.r.t. H) of class u_χ on X . Denote M_χ^s the stable locus of M_χ . Notice that when $\text{g.c.d}(\chi, L.H) = 1$, $M_\chi = M_\chi^s$.

Let $|L|^{IC}$ be the open subset of $|L|$ consisting of points corresponding to integral curves. By (\mathbf{A}'_2) , we have $|L| - |L|^{IC}$ is of codimension ≥ 2 in $|L|$.

There is a projection $\pi_\chi : M_\chi \rightarrow |L|$ which is defined by sending every sheaf to its schematic support. π_χ is a morphism according to Proposition 3.0.2 in [8]. (\mathbf{A}'_1) implies that $\text{Ext}^2(\mathcal{F}, \mathcal{F}) = 0$ for all \mathcal{F} semistable of class u_χ that are supported on integral curves. Hence by Lemma 4.2.3 in [8] the moduli space M_χ is smooth of dimension $g_L + l$ at the point $[\mathcal{F}]$ if \mathcal{F} is supported on an integral curve, i.e. $\pi_\chi([\mathcal{F}]) \in |L|^{IC}$.

For $\chi = 0$ we write u , M , M^s and π instead. It is easy to see that M does not depend on the polarization, but M_χ might for $\chi \neq 0$.

We denote Θ and λ_{pt} the determinant line bundles on $M_X^H(u)$ associated to $[\mathcal{O}_X]$ and $[\mathcal{O}_{pt}]$. Hence we have $\lambda_{c_n} \simeq \Theta^{\otimes r} \otimes \lambda_{pt}^{\otimes -n}$. We moreover ask \mathcal{O}_{pt} not to be supported at the base point of $|L|$, then by Proposition 2.8 in [6] we have that $\lambda_{pt} \simeq \pi^*\mathcal{O}_{|L|}(-1)$. Let $\Theta^r(n) := \Theta^r \otimes \pi^*\mathcal{O}_{|L|}(n)$.

3 Restrict Θ^r to intersections of pull backs of hyperplanes in $|L|$.

Choose $l - 1$ generic points in X : x_1, x_2, \dots, x_{l-1} . For each x_i , by asking the supporting curves of the sheaves to pass through it, we can get an equation f_i up to scalar in $|\pi_\chi^*\mathcal{O}_{|L|}(1)|$. Let V_i be the divisor defined by f_i . Since x_1, \dots, x_{l-1} are generic, we let V_i intersect each other transversally. There is

also a series of closed subschemes in $|L|$: P_1, P_2, \dots, P_{l-1} , where P_i consists of curves passing through x_1, \dots, x_i . $P_i \simeq \mathbb{P}^{l-i}$ and $\pi_\chi^{-1}(P_i) = \cap_{1 \leq m \leq i} V_m$. Let $T := P_{l-1}$. Then T is a projective line in $|L|$.

Because $|L| - |L|^{IC}$ is of codimension ≥ 2 in $|L|$, we can assume that $T \subset |L|^{IC}$. We then have the following Cartesian diagram

$$\begin{array}{ccccc} M_\chi^T & \xrightarrow{s} & M_\chi^{IC} & \xrightarrow{j} & M_\chi \\ \pi_\chi^T \downarrow & & \downarrow \pi_\chi^{IC} & & \downarrow \pi_\chi \\ T & \xrightarrow{t} & |L|^{IC} & \xrightarrow{i} & |L| \end{array} \quad (3.1)$$

M_χ^{IC} is contained in the stable locus M_χ^s and is smooth. We can also assume that M_χ^T is smooth since $|\pi_\chi^* \mathcal{O}_{|L|}(1)|$ has no base point.

For $\chi = 0$, $M_\chi = M$, we have an exact sequence on M :

$$0 \rightarrow \pi_\chi^* \mathcal{O}_{|L|}(-1) \rightarrow \mathcal{O}_M \rightarrow \mathcal{O}_{\pi^{-1}(P_1)} \rightarrow 0. \quad (3.2)$$

We then tensor (3.2) by $\Theta^r(n)$

$$0 \longrightarrow \Theta^r(n-1) \longrightarrow \Theta^r(n) \longrightarrow \Theta^r(n)|_{\pi^{-1}(P_1)} \longrightarrow 0. \quad (3.3)$$

Taking the global sections, we have

$$0 \rightarrow H^0(\Theta^r(n-1)) \rightarrow H^0(\Theta^r(n)) \rightarrow H^0(\Theta^r(n)|_{\pi^{-1}(P_1)}) \rightarrow H^1(\Theta^r(n-1)). \quad (3.4)$$

Sequence (3.4) implies that $h^0(\Theta^r(n)) - h^0(\Theta^r(n-1)) \leq h^0(\Theta^r(n)|_{\pi^{-1}(P_1)})$.

Denote $Z_i^r(t) = \sum_n h^0(M, \Theta^r(n)|_{\pi^{-1}(P_i)}) t^n$ for all $i = 1, \dots, l-1$. Notice that the sum are bounded from below for all i . Hence we have

$$h^0(M, \Theta^r(n)) \leq \sum_{m \leq n} h^0(\Theta^r(m)|_{\pi^{-1}(P_1)}) \quad (3.5)$$

The inequality (3.5) will become an equality if $h^1(M, \Theta^r(n-1)) = 0$ for all n such that $h^0(\pi^{-1}(P_1), \Theta^r(n)|_{\pi^{-1}(P_1)}) \neq 0$. And once the strict inequality holds for $n = n_0$, it holds for all $n \geq n_0$. On the other hand we have

$$\sum_n \left(\sum_{m \leq n} h^0(\Theta^r(m)|_{\pi^{-1}(P_1)}) \right) t^n = \frac{Z_1^r(t)}{1-t}.$$

Inductively for all $1 \leq i \leq l-2$, we have an exact sequence

$$0 \rightarrow \Theta^r(n-1)|_{\pi^{-1}(P_i)} \rightarrow \Theta^r(n)|_{\pi^{-1}(P_i)} \rightarrow \Theta^r(n)|_{\pi^{-1}(P_{i+1})} \rightarrow 0, \quad (3.6)$$

This implies that

$$h^0(M, \Theta^r(n)|_{\pi^{-1}(P_i)}) \leq \sum_{m \leq n} h^0(\Theta^r(n)|_{\pi^{-1}(P_{i+1})}) \quad (3.7)$$

Finally we come to the generic projective line $T = P_{l-1}$ in the linear system. Define

$$\sum_n a_n^r t^n := \frac{Z_{l-1}^r(t)}{(1-t)^{l+1}}.$$

Then we have

$$h^0(M, \Theta^r(n)) \leq a_n^r. \quad (3.8)$$

We will compute $Z_{l-1}^r(t)$ for $g_L = 1, 2$ in the next sections.

4 Moduli spaces over one dimensional linear systems.

In this section, we construct a new moduli space \tilde{M}_χ over a one dimensional linear system $|\tilde{L}|$ on a surface \tilde{X} obtained by blowing up points in X . Then we show that \tilde{M}_χ can be identified with M_χ^T . The construction is as follows.

Choose $l-1$ generic points in X : x_1, x_2, \dots, x_{l-1} ; such that curves passing through all these $l-1$ points are integral curves (this is to say that the line T defined by those points is contained in $|L|^{IC}$) and all of them except finitely many are smooth. Moreover those curves are smooth at x_1, x_2, \dots, x_{l-1} (this is possible since the points are finitely many). We then blow up all these $l-1$ points and get a new surface \tilde{X} together with a projection $\rho : \tilde{X} \rightarrow X$. We have a new moduli space $\tilde{M}_\chi = M_{\tilde{X}}(\tilde{u}_\chi)$, where $\tilde{u}_\chi = (0, \tilde{L} = \rho^*L - E_1 - E_2 - \dots - E_{l-1}, \chi)$ with the E_i the exceptional divisors. Notice that there is a natural closed embedding $\iota : |\tilde{L}| \rightarrow |L|$ with its image T . In particular for $\tilde{u}_\chi = \tilde{u}_0 =: \tilde{u}$, we denote $\tilde{\Theta}$ the determinant line bundle on $\tilde{M} = \tilde{M}_0$ associated to the structure sheaf $\mathcal{O}_{\tilde{X}}$. Then we have the following proposition:

Proposition 4.1. *There is a morphism $\underline{f} : \tilde{M}_\chi \rightarrow M_\chi$, which factors through the embedding $j \circ s$ as in diagram (3.1) and induces an isomorphism $f : \tilde{M}_\chi \rightarrow$*

M_χ^T ; and we have the Cartesian diagram as follows

$$\begin{array}{ccccc}
\tilde{M}_\chi & \xrightarrow{f} & M_\chi^T & \xrightarrow{j \circ s} & M_\chi \\
\tilde{\pi}_\chi \downarrow & & \pi_\chi^T \downarrow & & \downarrow \pi_\chi \\
|\tilde{L}| & \xrightarrow{i} & T & \xrightarrow{i \circ t} & |L|
\end{array} \tag{4.1}$$

And moreover for $\chi = 0$ we have $\underline{f}_* \tilde{\Theta}^r \simeq (j \circ s)_* f_* \tilde{\Theta}^r$ and $f_* \tilde{\Theta}^r \simeq (j \circ s)^* \Theta^r$.

Proof. First we have two lemmas

Lemma 4.2. *There is a universal sheaf on $\tilde{X} \times \tilde{M}_\chi$. That is to say, \tilde{M}_χ is a fine moduli space.*

Proof. Let $\tilde{\Omega}_\chi$ be the open subscheme of the Quot-scheme and $\tilde{\phi}_\chi : \tilde{\Omega}_\chi \rightarrow \tilde{M}_\chi$ be the good quotient. Since all curves in $|\tilde{L}|$ are irreducible and reduced, all semistable sheaves in \tilde{u}_χ are stable and the morphism $\tilde{\phi}_\chi : \tilde{\Omega}_\chi \rightarrow \tilde{M}_\chi$ is a principal G -bundle, with G some reductive group. There is a universal quotient $\tilde{\mathcal{E}}_\chi$ on $\tilde{X} \times \tilde{\Omega}_\chi$.

$$\begin{array}{ccc}
\tilde{\mathcal{E}}_\chi & \longrightarrow & \tilde{X} \times \tilde{\Omega}_\chi \\
& \searrow q & \downarrow p_\chi \\
& & \tilde{\Omega}_\chi
\end{array}$$

Let $A = \det R^\bullet p_{\chi*}(\tilde{\mathcal{E}}_\chi \otimes q^* \mathcal{O}_{\tilde{X}}((1-\chi)E_1))$. A is a line bundle on $\tilde{\Omega}_\chi$ and carries a natural G -linearization of Z -weight $\chi((\tilde{\mathcal{E}}_\chi)_y \otimes \mathcal{O}_{\tilde{X}}((1-\chi)E_1))$ for every closed point $y \in \tilde{\Omega}_\chi$. Since $E_i \cdot \tilde{L} = 1$ and $(\tilde{\mathcal{E}}_\chi)_y$ is of rank 0 and Euler characteristic χ for every y , we have $\chi((\tilde{\mathcal{E}}_\chi)_y \otimes \mathcal{O}_{\tilde{X}}((1-\chi)E_1)) = 1$ which means A is of Z -weight 1. According to Proposition 4.6.2 and Theorem 4.6.5 in [5], we have the lemma. \square

Lemma 4.3. *$\tilde{\pi}$ is flat and \tilde{M}_χ is an integral scheme.*

Proof. Since curves in $|\tilde{L}|$ are reduced and irreducible and with at most planar singularities, every fiber of $\tilde{\pi}$ is integral and of dimension g . Hence \tilde{M}_χ can not have more than one component because $|\tilde{L}|$ is just a projective line. Then $\tilde{\pi}$ is flat because there is no component contained in any fiber. \tilde{M}_χ is reduced because all fibers of $\tilde{\pi}$ are reduced and $|\tilde{L}|$ is reduced. \square

Now let $\tilde{\mathcal{U}}_\chi$ be a universal sheaf on $\tilde{X} \times \tilde{M}_\chi$. Push it forward along $\rho \times id_{\tilde{M}_\chi}$ and get a flat family $\mathcal{U}_\chi := (\rho \times id_{\tilde{M}_\chi})_* \tilde{\mathcal{U}}_\chi$ on $X \times \tilde{M}_\chi$.

Over every point $[\mathcal{F}] \in \tilde{M}_\chi$, $\rho_* \mathcal{F}$ is a stable sheaf whose support is the push forward of the support of \mathcal{F} , hence $[\rho_* \mathcal{F}] \in M_\chi^T$. The flat family \mathcal{U}_χ induces a morphism $\underline{f} : \tilde{M}_\chi \rightarrow M_\chi^T$, with its image contained in M_χ^T .

Since M_χ^T is smooth hence normal and \tilde{M}_χ is integral, to prove that $f : \tilde{M}_\chi \rightarrow M_\chi^T$ is an isomorphism, it is enough to show that it is bijective. The injectivity is because $\rho|_{C_\mathcal{F}} : C_\mathcal{F} \rightarrow C_{\rho_* \mathcal{F}}$ is an isomorphism, where $C_\mathcal{F}$ is the supporting curve of \mathcal{F} . To prove the surjectivity, we need to show that $\forall [\mathcal{G}] \in M_\chi^T$, $\exists [\tilde{\mathcal{G}}] \in \tilde{M}_\chi$ such that $\rho_* \tilde{\mathcal{G}} \simeq \mathcal{G}$. Pull back \mathcal{G} to get a sheaf on \tilde{X} with support $C = C_{\rho_* \mathcal{G}} \in |\rho^* L|$. On \tilde{X} we have

$$0 \rightarrow \mathcal{O}_{E_i}(-1)^{\oplus_{i=1}^{l-1}} \rightarrow \mathcal{O}_C \rightarrow \mathcal{O}_{\tilde{C}} \rightarrow 0.$$

Tensor this sequence by $\rho^* \mathcal{G}$.

$$Tor^1(\rho^* \mathcal{G}, \mathcal{O}_{\tilde{C}}) \xrightarrow{\tau} \mathcal{O}_{E_i}(-1)^{\oplus_{i=1}^{l-1}} \otimes \rho^* \mathcal{G} \longrightarrow \rho^* \mathcal{G} \longrightarrow \mathcal{O}_{\tilde{C}} \otimes \rho^* \mathcal{G} \longrightarrow 0.$$

$c_1(\mathcal{O}_{\tilde{C}} \otimes \rho^* \mathcal{G}) = \tilde{L}$, so $c_1(\text{im} \tau) = 0$, while $\text{im} \tau$ (i.e. the image of τ) is contained in $\mathcal{O}_{E_i}(-1)^{\oplus_{i=1}^{l-1}} \otimes \rho^* \mathcal{G} = \mathcal{O}_{E_i}(-1)^{\oplus_{i=1}^{l-1}}$, which is pure on its support. Therefore $\tau = 0$. Hence we have

$$0 \rightarrow \mathcal{O}_{E_i}(-1)^{\oplus_{i=1}^{l-1}} \rightarrow \rho^* \mathcal{G} \rightarrow \mathcal{O}_{\tilde{C}} \otimes \rho^* \mathcal{G} \rightarrow 0.$$

Push it forward. Because of the vanishing of $\rho_* \mathcal{O}_{E_i}(-1)$ and $R^1 \rho_* \mathcal{O}_{E_i}(-1)$, we have $\rho_*(\rho^* \mathcal{G}) \simeq \rho_*(\mathcal{O}_{\tilde{C}} \otimes \rho^* \mathcal{G})$.

ρ restricted on \tilde{C} is an isomorphism. So if $\rho_*(\rho^* \mathcal{G}) \simeq \mathcal{G}$, then $\mathcal{O}_{\tilde{C}} \otimes \rho^* \mathcal{G}$ is a pure sheaf of rank 1 on \tilde{C} and of Euler characteristic 0, hence $[\mathcal{O}_{\tilde{C}} \otimes \rho^* \mathcal{G}] \in \tilde{M}_\chi$, and hence we have found $[\tilde{\mathcal{G}}] = [\mathcal{O}_{\tilde{C}} \otimes \rho^* \mathcal{G}] \in \tilde{M}_\chi$, such that $f([\tilde{\mathcal{G}}]) = [\mathcal{G}]$.

Now we only need to show $\rho_*(\rho^* \mathcal{G}) \simeq \mathcal{G}$. Firstly, we show that $\rho_*(\rho^* \mathcal{O}_C) \simeq \mathcal{O}_C$. This can be seen from $\rho_*(\rho^* \mathcal{G}) \simeq \rho_*(\mathcal{O}_{\tilde{C}} \otimes \rho^* \mathcal{G})$, with $\mathcal{G} = \mathcal{O}_C$. Then since \mathcal{G} is locally free on its support outside the singular points, we have that the isomorphism holds outside the singular points; but around the singular points, ρ is an isomorphism.

Finally let $\chi = 0$. The claim on the determinant line bundles is somehow obvious: by the universal property of Θ , we have $\underline{f}^*(\Theta) \simeq (\det R^\bullet p \mathcal{U})^\vee$, where

\mathcal{U} is the flat family on $X \times \tilde{M}$ obtained by pushing $\tilde{\mathcal{U}}$ forward along $\rho \times id_{\tilde{M}}$.

$$\begin{array}{ccc} \tilde{\mathcal{U}} & \longrightarrow & \tilde{X} \times \tilde{M} \\ \downarrow (\rho \times id_{\tilde{M}})_* & & \downarrow \rho \times id_{\tilde{M}} \\ \mathcal{U} & \longrightarrow & X \times \tilde{M} \\ & & \downarrow p \\ & & \tilde{M} \end{array}$$

Hence $R^\bullet p \mathcal{U} \simeq R^\bullet p ((\rho \times id_{\tilde{M}})_* \tilde{\mathcal{U}})$.

Lemma 4.4. $R^i(\rho \times id_{\tilde{M}})_* \tilde{\mathcal{U}} = 0$, for all $i > 0$.

Proof. One can see that $\rho \times id_{\tilde{M}}$ is an isomorphism when restricted to the support of $\tilde{\mathcal{U}}$, hence the lemma. \square

As $R^i(\rho \times id_{\tilde{M}})_* \tilde{\mathcal{U}} = 0$, for all $i > 0$, we have $\underline{f}^* \Theta = \det R^\bullet p \mathcal{U} \simeq \det R^\bullet (p \circ (\rho \times id_{\tilde{M}})) \tilde{\mathcal{U}} = \tilde{\Theta}$. Hence $f_*(\tilde{\Theta}^r) \simeq f_*(\underline{f}^*(\Theta^r)) \simeq \underline{f}_*(\mathcal{O}_{\tilde{M}}) \otimes \Theta^r \simeq (j \circ s)_* \mathcal{O}_{M^T} \otimes \Theta^r$ and $f_* \tilde{\Theta}^r \simeq (j \circ s)^* \Theta^r$ for all r . So we have proven the proposition. \square

Remark 4.5. According to Proposition 4.1, \tilde{M}_χ is a smooth projective scheme of dimension $g_L + 1$. But $Ext^2(\mathcal{F}, \mathcal{F})_0$ may not vanish for $[\mathcal{F}] \in \tilde{M}$, because (\tilde{L}, \tilde{K}) might not satisfy (\mathbf{A}'_1) .

Remark 4.6. For the moduli space \tilde{M}_χ , we did not specify the ample line bundle $\mathcal{O}_{\tilde{X}}(1)$ on the blow-up \tilde{X} , but it is easy to see that the moduli space \tilde{M}_χ does not depend on the polarization.

Proposition 4.7. \tilde{M}_χ is isomorphic to \tilde{M} for any $\chi \in \mathbb{Z}$.

Proof. Recall that \tilde{M}_χ is a fine moduli space for any χ . Let $\tilde{\mathcal{U}}_\chi$ be some universal sheaf on $\tilde{X} \times \tilde{M}_\chi$. We have the diagram

$$\begin{array}{ccc} \tilde{\mathcal{U}}_\chi & \longrightarrow & \tilde{X} \times \tilde{M}_\chi \\ & \searrow q & \downarrow p_\chi \\ & \tilde{X} & \tilde{M}_\chi \end{array}$$

Then $\tilde{\mathcal{U}}_\chi \otimes q^* \mathcal{O}_{\tilde{X}}((-\chi)E_1)$ is a flat family on $\tilde{X} \times \tilde{M}_\chi$ of stable sheaves of class \tilde{u} , and hence induces a morphism $\varphi_\chi : \tilde{M}_\chi \rightarrow \tilde{M}$. It is easy to see

that φ_χ is bijective, hence an isomorphism since both \tilde{M}_χ and \tilde{M} are smooth. Notice that one can construct the isomorphism φ_χ in many ways and there is no canonical way if $l \geq 2$. \square

Now we have identified $(\tilde{M}, \tilde{\Theta}^r)$ with $(M^T, \Theta^r|_{M^T})$, hence we can focus on $\tilde{\pi}_* \tilde{\Theta}^r$ on $|\tilde{L}|$, instead of $\pi_*^T(\Theta^r|_{M^T})$ on T .

Lemma 4.8. (1) $R^i \tilde{\pi}_* \tilde{\Theta}^r = 0$ for all $i > 0$ and $r > 0$, $R^i \tilde{\pi}_* \tilde{\Theta}^r = 0$ for all $i < g_L$ and $r < 0$;

(2) For $r > 0$, $\tilde{\pi}_* \tilde{\Theta}^r$ is locally free of rank r^{g_L} and $\tilde{\pi}_* \tilde{\Theta} \simeq \mathcal{O}_{|\tilde{L}|}$;

(3) For $r < 0$, $R^{g_L} \tilde{\pi}_* \tilde{\Theta}^r$ is locally free of rank $(-r)^{g_L}$.

Proof. By Proposition 3.0.4 in [8] we know that $\tilde{\Theta}(s)$ is ample for $s \gg 0$, hence $\tilde{\Theta}$ restricted to every fiber of $\tilde{\pi}$ is ample. By Corollary 4.12 that we will prove later, the dualizing sheaf on every fiber of $\tilde{\pi}$ is invertible and corresponds to a torsion class in the Picard group. Hence restricted to every fiber $\tilde{\Theta}^r$ has no higher cohomology for $r > 0$. Hence $R^i \tilde{\pi}_* \tilde{\Theta}^r = 0$ for all $i > 0$ and $r > 0$ and $\tilde{\pi}_* \tilde{\Theta}^r$ is locally free. Moreover by the basic theory of Jacobians, we know that $\tilde{\pi}_* \tilde{\Theta}^r$ is of rank r^{g_L} . When $r = 1$, $\tilde{\pi}_* \tilde{\Theta}$ is a line bundle with a nowhere vanishing section hence isomorphic to $\mathcal{O}_{|\tilde{L}|}$.

The argument for $r < 0$ is analogous. \square

Proof of Theorem 1.1. From the result in [4], we know that

$$Y^1(t) = \sum_n h^0(M(c_n^1), \lambda_u) t^n = \frac{1}{(1-t)^{l+1}}.$$

Then Theorem 1.1 is just a corollary of the Statement 2 in Lemma 4.8. \square

We obtain the moduli space \tilde{M} by blowing up $l - 1$ generic points x_1, \dots, x_{l-1} on X . On the other hand we may first blow up one point x_1 to get a surface X_1 with the morphism $\rho_1 : X_1 \rightarrow X$, and let $L_1 = \rho_1^* L - E_1$. Then similarly we have the moduli space M_1 and Θ_1 which is the determinant line bundle associated to \mathcal{O}_{X_1} . Tautologically, blowing up the $l - 1$ points x_1, \dots, x_{l-1} in X is the same as blowing up $\rho_1(x_2), \dots, \rho_1(x_{l-1})$ in X_1 . Hence we get the same triple $(\tilde{X}, \tilde{M}, \tilde{\Theta})$ for both (X, M, Θ) and (X_1, M_1, Θ_1) . There is a rational map $\nu : M_1 \dashrightarrow M$, but not necessary a morphism in general. However because of Proposition 4.1, we have the following trivial remark. Notice that if L satisfies condition (A'_2) , then so does L_1 for x_1 generic. And $K.L = K_1.L_1 - 1$ with $K_1 = \rho_1^* K + E_1$ the canonical divisor on X_1 .

Remark 4.9. Let (X, M, Θ) , (X_1, M_1, Θ_1) and $(\tilde{X}, \tilde{M}, \tilde{\Theta})$ be as in the previous paragraph. Let T be the projective line in $|L|$ defined by asking curves to pass through all the $l - 1$ points x_1, \dots, x_{l-1} , and T_1 the line in $|L_1|$ consisting of curves passing through all the $l - 2$ points $\rho_1(x_2), \dots, \rho_1(x_{l-1})$. If L satisfies (A'_1) and $L.K < -1$, then we have the following Cartesian diagram with f and f_1 isomorphisms and $f^*\Theta^r \simeq f_1^*\Theta_1^r \simeq \tilde{\Theta}^r$.

$$\begin{array}{ccccccc}
M_1 & \xleftarrow{j_1 \circ s_1} & M_1^{T_1} & \xleftarrow{f_1} & \tilde{M} & \xrightarrow{f} & M^T \xrightarrow{j \circ s} M \\
\pi_1 \downarrow & & \pi_1^{T_1} \downarrow & & \downarrow \tilde{\pi}_\chi & & \downarrow \pi^T & \downarrow \pi \\
|L_1| & \xleftarrow{i_1 \circ t_1} & T_1 & \xleftarrow{i_1} & |\tilde{L}| & \xrightarrow{i} & T & \xrightarrow{i \circ t} |L|
\end{array} \tag{4.2}$$

For M_χ with any χ , we have an analogous Cartesian diagram as (4.2).

At the end of this section, we prove some lemmas which will be used in the next two sections. Let (X, L) and (\tilde{X}, \tilde{L}) be the same as in Proposition 4.1. K and \tilde{K} are the canonical divisor on X and \tilde{X} respectively, and $\tilde{K} = \rho^*K + E_1 + \dots + E_{l-1}$. Since there is more than one integral curve in $|L|$, (A'_1) implies that K is not effective, hence nor is \tilde{K} .

Lemma 4.10. $h^1(\tilde{L}) = h^1(L) = 0$, $h^2(\tilde{L}) = h^2(L) = 0$, hence $\chi(L) = l + 1$ and $\chi(\tilde{L}) = 2$.

Proof. Since K is noneffective, $L^{-1} \otimes K$ must be noneffective which means $h^0(L^{-1} \otimes K) = h^2(L) = 0$. Similarly $h^2(\tilde{L})$ must be zero because \tilde{K} is not effective. By a direct computation we get $\chi(L) - \chi(\tilde{L}) = h^0(L) - h^0(\tilde{L}) = l - 1$, hence $h^1(L) = h^1(\tilde{L})$.

On X we have the following exact sequence

$$0 \rightarrow L^{-1} \otimes K \rightarrow K \rightarrow \mathcal{O}_C(K) \rightarrow 0,$$

with C some smooth curve in $|L|$. $L.K < 0$, hence $\mathcal{O}_C(K)$ is locally free on C with negative degree and has no sections. So there is an injective map sending $H^1(L^{-1} \otimes K)$ into $H^1(K)$. So $h^1(L) = h^1(L^{-1} \otimes K) \leq h^1(K)$. X is simply connected, then $H^1(K) = 0$ and $h^1(L) = 0$. Hence the lemma. \square

Lemma 4.11. Let $\omega_{M_\chi^{IC}}$ denote the canonical line bundle of M_χ^{IC} , then we have $c_1(\omega_{M_\chi^{IC}}) = [(\pi_\chi^{IC})^* \mathcal{O}_{|L|^{IC}}(1)^{\otimes L.K}]$.

Proof. The proof is essentially the same as what Danila does in [3] for $X = \mathbb{P}^2$. M_χ^{IC} is smooth. Hence it will suffice to prove that $c_1(\mathcal{T}_{M_\chi^{IC}}) = [(\pi_\chi^{IC})^* \mathcal{O}_{|L|}(-1)^{\otimes L.K}]$, where $\mathcal{T}_{M_\chi^{IC}}$ is the tangent bundle on M_χ^{IC} .

Recall there is a morphism $\phi_\chi^{IC} : \Omega_\chi^{IC} \rightarrow M_\chi^{IC}$ which is a principal G -bundle with $G = PGL(V)$. We have $Pic(M_\chi^{IC}) \simeq Pic^G(\Omega_\chi^{IC})$ (Theorem 4.2.16 in [5]). And also because there is no surjective homomorphism from G to \mathbb{G}_m , the natural morphism $Pic^G(\Omega_\chi^{IC}) \rightarrow Pic(\Omega_\chi^{IC})$ is injective ([7] Chap 1, Section 3, Proposition 1.4). Hence it is enough to prove that $(\phi_\chi^{IC})^*(c_1(\mathcal{T}_{M_\chi^{IC}})) = [(\phi_\chi^{IC})^*(\pi_\chi^{IC})^*\mathcal{O}_{|L|}(-1)^{\otimes L \cdot K}]$

We have a universal sheaf on $X \times \Omega_\chi^{IC}$. We denote it \mathcal{E}_χ^{IC} .

$$\begin{array}{ccc}
 \mathcal{E}_\chi^{IC} & \longrightarrow & X \times \Omega_\chi^{IC} \\
 & \searrow q & \downarrow p_\chi \\
 & X & \Omega_\chi^{IC} \\
 & & \downarrow \phi_\chi^{IC} \\
 & & M_\chi^{IC} \\
 & & \downarrow \pi_\chi^{IC} \\
 & & |L|^{IC}
 \end{array} \tag{4.3}$$

In the Grothendieck group, we have

$$(\phi_\chi^{IC})^*\mathcal{T}_{M_\chi^{IC}} = \mathcal{E}xt_{p_\chi}^1(\mathcal{E}_\chi^{IC}, \mathcal{E}_\chi^{IC}).$$

And $(\phi_\chi^{IC})^*(c_1(\mathcal{T}_{M_\chi^{IC}})) = c_1((\phi_\chi^{IC})^*\mathcal{T}_{M_\chi^{IC}})$. So it is enough to compute $c_1((\phi_\chi^{IC})^*\mathcal{T}_{M_\chi^{IC}})$.

Because of (\mathbf{A}'_1) , we have that over every closed point $y \in \Omega_\chi^{IC}$, $\text{Ext}^i((\mathcal{E}_\chi^{IC})_y, (\mathcal{E}_\chi^{IC})_y) = 0$, for all $i \geq 2$. Hence $\mathcal{E}xt_{p_\chi}^i(\mathcal{E}_\chi^{IC}, \mathcal{E}_\chi^{IC}) = 0$, for all $i \geq 2$, because fiberwise they are $\text{Ext}^i((\mathcal{E}_\chi^{IC})_y, (\mathcal{E}_\chi^{IC})_y)$. Also we have $\text{Ext}^0((\mathcal{E}_\chi^{IC})_y, (\mathcal{E}_\chi^{IC})_y) = \mathbb{C}$, hence $\mathcal{E}xt_{p_\chi}^0(\mathcal{E}_\chi^{IC}, \mathcal{E}_\chi^{IC}) = (p_\chi)_*\mathcal{H}om(\mathcal{E}_\chi^{IC}, \mathcal{E}_\chi^{IC})$ is a line bundle on Ω_χ^{IC} , hence isomorphic to $\mathcal{O}_{\Omega_\chi^{IC}}$ since it has a nowhere vanishing global section. Therefore

$$[\det \mathcal{E}xt_{p_\chi}^\bullet(\mathcal{E}_\chi^{IC}, \mathcal{E}_\chi^{IC})] = [\det R^\bullet p_\chi(\mathcal{E}xt^\bullet(\mathcal{E}_\chi^{IC}, \mathcal{E}_\chi^{IC}))] = [(\det \mathcal{E}xt_{p_\chi}^1(\mathcal{E}_\chi^{IC}, \mathcal{E}_\chi^{IC}))^\vee].$$

Hence

$$c_1((\phi_\chi^{IC})^*\mathcal{T}_{M_\chi^{IC}}) = -c_1(\det R^\bullet p_\chi(\mathcal{E}xt^\bullet(\mathcal{E}_\chi^{IC}, \mathcal{E}_\chi^{IC}))) = -c_1(R^\bullet p_\chi(\mathcal{E}xt^\bullet(\mathcal{E}_\chi^{IC}, \mathcal{E}_\chi^{IC}))). \tag{4.4}$$

By Grothendieck-Riemann-Roch,

$$ch(R^\bullet p_\chi \mathcal{E}xt^\bullet(\mathcal{E}_\chi^{IC}, \mathcal{E}_\chi^{IC})) = (p_\chi)_*(ch(\mathcal{E}_\chi^{IC}) \cdot ch((\mathcal{E}_\chi^{IC})^\vee) \cdot td(q^*\mathcal{T}_X)),$$

where \mathcal{T}_X is the tangent sheaf on X .

Since \mathcal{E}_χ^{IC} is a torsion sheaf on $X \times \Omega_\chi^{IC}$,

$$c_1(R^\bullet p_\chi \mathcal{E}xt^\bullet(\mathcal{E}_\chi^{IC}, \mathcal{E}_\chi^{IC})) = (p_\chi)_* \left(-\frac{1}{2} c_1(\mathcal{E}_\chi^{IC}) c_1(\mathcal{E}_\chi^{IC}) c_1(q^* \mathcal{T}_X) \right) = (p_\chi)_* \left(\frac{1}{2} c_1(\mathcal{E}_\chi^{IC})^2 c_1(q^* K) \right). \quad (4.5)$$

$c_1(\mathcal{E}_\chi^{IC})$ is just the support of \mathcal{E}_χ^{IC} , which is the pull back along $id_X \times (\pi_\chi^{IC} \circ \phi_\chi^{IC})$ of the universal curve in $X \times |L|^{IC}$. Therefore, $c_1(\mathcal{E}_\chi^{IC}) = q^* L \otimes p_\chi^* F$, where F is the fiber class of π_χ^{IC} in Ω_χ^{IC} , i.e. $\mathcal{O}_{\Omega_\chi^{IC}}(F) \simeq (\phi_\chi^{IC})^* \circ (\pi_\chi^{IC})^* \mathcal{O}_{|L|}(1)$. Since $q^* L \cdot q^* L \cdot q^* K = 0$, so we have

$$\frac{1}{2} (c_1(\mathcal{E}_\chi^{IC}))^2 \cdot (q^* K) = q^* L \cdot q^* K \cdot p^* F + \frac{1}{2} q^* K \cdot (p_\chi^* F)^2.$$

and also $(p_\chi)_*(q^* K \cdot (p_\chi^* F)^2) = 0$, so

$$\begin{aligned} (p_\chi)_* \left(\frac{1}{2} (c_1(\mathcal{E}_\chi^{IC}))^2 \cdot (q^* K) \right) &= (p_\chi)_* (q^* L \cdot q^* K \cdot p_\chi^* F) \\ &= (L \cdot K) F. \end{aligned}$$

Hence together with (4.4) and (4.5) we have

$$c_1((\phi_\chi^{IC})^* \mathcal{T}_{M_\chi^{IC}}) = [(\phi_\chi^{IC})^* (\pi_\chi^{IC})^* \mathcal{O}_{|L|^{IC}}(-1)^{\otimes L \cdot K}].$$

Hence the lemma. \square

Corollary 4.12. $c_1(\mathcal{T}_{\tilde{M}}) = [\tilde{\pi}^* \mathcal{O}_{|\tilde{L}|}(-1)^{\otimes (g_L - 2)}]$, where $\mathcal{T}_{\tilde{M}}$ is the tangent bundle on \tilde{M} .

Proof. Since \tilde{M} is smooth, $c_1(\mathcal{T}_{\tilde{M}}) = -c_1(\omega_{\tilde{M}})$, where $\omega_{\tilde{M}}$ is the canonical line bundle on \tilde{M} . Moreover as stated in Proposition 4.1, $\omega_{\tilde{M}} = f^* \omega_{M^T}$. Because M^T is a complete intersection of $l - 1$ divisors in $|\pi^* \mathcal{O}_{|L|}(1)|$ in M^{IC} and also because of Lemma 4.11, we have $c_1(\omega_{M^T}) = [(\pi^T)^* \mathcal{O}_T(L \cdot K + l - 1)]$ and hence $c_1(\omega_{\tilde{M}}) = [f^*(\pi^T)^* \mathcal{O}_T(L \cdot K + l - 1)] = [\tilde{\pi}^* \mathcal{O}_{|\tilde{L}|}(L \cdot K + l - 1)]$. Since $L \cdot K + l - 1 = g_L - 2 + h^1(L) - h^1(K) = g_L - 2$, we have the lemma. \square

5 Splitting type for genus one case.

From now on we are always working on \tilde{M} . So for simplicity, we drop all the \sim and just write X, L, M, Θ^r, π , etc.

Now M is a flat family of Jacobians over $|L| \simeq \mathbb{P}^1$. We will give the formulas for $g_L = 1, 2$ by giving the explicit splitting types for all $\pi_* \Theta^r$, $r > 0$.

By Lemma 3.0.1 in [8], there is a natural global section of Θ which vanishes at $[\mathcal{F}] \in M$ such that $H^0(\mathcal{F}) \neq 0$. Let $D_\Theta = \{[\mathcal{F}] \in M : h^0(\mathcal{F}) \neq 0\}$ be the divisor associated to that section.

We prove the following proposition in this section. The technique we use is essentially the same as that in [8] for genus one case.

Proposition 5.1. *If $g_L = 1$, then for $r \geq 2$,*

$$\pi_* \Theta^r \simeq \mathcal{O}_{|L|} \oplus (\mathcal{O}_{|L|}(-i))^{\oplus_{i=2}^r}.$$

Proof. In $X \times |L| \simeq X \times \mathbb{P}^1$, there is a universal curve \mathcal{C} such that every fiber \mathcal{C}_s is just the curve represented by $s \in |L|$.

$$\begin{array}{ccc} \mathcal{C} & \longrightarrow & X \times |L| \\ & \searrow q & \downarrow p \\ X & & |L| \end{array}$$

Since \mathcal{C}_s is integral of genus one, $\mathcal{O}_{\mathcal{C}_s}$ is stable of Euler characteristic zero for every s . Hence the structure sheaf $\mathcal{O}_{\mathcal{C}}$ of \mathcal{C} induces an injective morphism embedding $|L|$ as a subscheme of M .

$$\iota : |L| \rightarrow M.$$

It is easy to see that ι provides a section of the projection π . The image of ι is contained in D_Θ , and moreover we have the following lemma.

Lemma 5.2. *π restricted to D_Θ is an isomorphism and ι is its inverse.*

Proof. Let $[\mathcal{F}] \in M$, and C its support. Since C is integral and of genus one, we have $H^0(\mathcal{F}) \neq 0 \Leftrightarrow \mathcal{F} \simeq \mathcal{O}_C$. Hence D_Θ intersects every fiber of π at only one reduced point. Hence π restricted on it is a morphism of degree 1, hence an isomorphism. It is obvious to have $\iota \cdot \pi = id_{|L|}$. \square

Thus on M we have

$$0 \rightarrow \Theta^{-1} \rightarrow \mathcal{O}_M \rightarrow \mathcal{O}_{D_\Theta} \rightarrow 0.$$

Tensoring by Θ^r with $r \geq 2$, we get

$$0 \rightarrow \Theta^{r-1} \rightarrow \Theta^r \rightarrow \mathcal{O}_{D_\Theta}(\Theta^r) \rightarrow 0. \quad (5.1)$$

$R^1\pi_*\Theta^{r-1} = 0$ by Lemma 4.8. Push (5.1) forward via π and we have

$$0 \rightarrow \pi_*\Theta^{r-1} \rightarrow \pi_*\Theta^r \rightarrow \pi_*\mathcal{O}_{D_\Theta}(\Theta^r) \rightarrow 0. \quad (5.2)$$

Since $D_\Theta \simeq |L|$ and $\pi \cdot \iota = id_{|L|}$, $\pi_*\mathcal{O}_{D_\Theta}(\Theta^r) \simeq \pi_*\iota_*\iota^*\Theta^r \simeq \iota^*\Theta^r$.

According to the universal property of Θ , we have $\iota^*\Theta^r \simeq (det(R^\bullet p[\mathcal{O}_C]))^{-r}$.

We have an exact sequence on $X \times |L|$.

$$0 \rightarrow q^*\mathcal{O}_X(-L) \otimes p^*\mathcal{O}_{|L|}(-1) \rightarrow \mathcal{O}_{X \times |L|} \rightarrow \mathcal{O}_C \rightarrow 0.$$

Hence $(det(R^\bullet p[\mathcal{O}_C]))^{-1} \simeq (det(R^\bullet p[\mathcal{O}_{X \times |L|}]))^{-1} \otimes det(R^\bullet p[q^*\mathcal{O}_X(-L) \otimes p^*\mathcal{O}_{|L|}(-1)])$.

And also $det(R^\bullet p[\mathcal{O}_{X \times |L|}]) \simeq \mathcal{O}_{|L|}$; $det(R^\bullet p[q^*\mathcal{O}_X(-L) \otimes p^*\mathcal{O}_{|L|}(-1)]) \simeq \mathcal{O}_{|L|}(-1)^{\otimes \chi(\mathcal{O}_X(-L))}$.

Since $g_L = 1$, $\chi(\mathcal{O}_X(-L)) = \chi(\mathcal{O}_X) = 1$ and $\mathcal{O}_{|L|}(\Theta^r) \simeq \mathcal{O}_{|L|}(-r)$.

The exact sequence (5.2) splits for every $r > 1$. And by induction we get

$$\pi_*\Theta^r \simeq \mathcal{O}_{|L|} \oplus \mathcal{O}_{|L|}(-i)^{\oplus_{i=2}^r}.$$

□

In this case, the generating function can be written down as

$$\begin{aligned} Z^r(t) &= \sum_n h^0(M, \lambda_{c_n^r}) t^n \\ &= \sum_n h^0(M, \Theta^r \otimes \pi^*\mathcal{O}_{|L|}(n)) t^n \\ &= \sum_n h^0(|L|, \pi_*(\Theta^r) \otimes \mathcal{O}_{|L|}(n)) t^n \\ &= \frac{1 + t^2 + t^3 + \dots + t^r}{(1-t)^2}. \end{aligned}$$

Remark 5.3. *This result is compatible with Statement 2 in Theorem 4.4.1 in [8] as $X = \mathbb{P}^2$ and $L = 3H$ or $X = \mathbb{P}(\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(-e))$ and $L = 2G + (e+2)F$ with $e = 0, 1$.*

Proof of Theorem 1.2. Recall that we denote

$$Y_{g_L=1}^r(t) = \sum_{n \geq 0} y_{n, g_L=1}^r t^n = \frac{1 + t^2 + t^3 + \dots + t^r}{(1-t)^2};$$

and let $y_{n,g_L=1}^r = 0$ for all $n < 0$. In this case we have

$$Y_{g_L=1}^r(t) = \frac{Z^r(t)}{(1-t)^{l-1}},$$

hence Theorem 1.2. \square

6 Splitting type for genus two case.

Remember that we get the one-dimensional linear system $|L|$ by blowing up $l-1$ points. So we can write $L = L' - E_1 - \dots - E_{l-1}$ with L' effective and $E_i.L = 1$. We in addition ask $l \geq 3$. Then we have the following proposition.

Proposition 6.1. *For the one-dimensional linear system $L = L' - E_1 - \dots - E_{l-1}$ with $g_L = 2$, if $l-1 \geq 2$, then*

- (1) $\pi_*\Theta^{r-1}$ is a direct summand of $\pi_*\Theta^r$. Let $\pi_*\Theta^r = \pi_*\Theta^{r-1} \oplus \Delta_r$;
 - (2) $\pi_*\Theta^2 \simeq \mathcal{O}_{|L|} \oplus (\mathcal{O}_{|L|}(-2))^{\oplus 3}$, $\pi_*\Theta^3 \simeq \mathcal{O}_{|L|}(-4) \oplus (\mathcal{O}_{|L|}(-3))^{\oplus 4} \oplus (\mathcal{O}_{|L|}(-2))^{\oplus 3} \oplus \mathcal{O}_{|L|}$;
 - (3) for $r \geq 4$, we have the recursion formula
- $$\pi_*\Theta^r \simeq \pi_*\Theta^{r-1} \oplus (\mathcal{O}_{|L|}(-r))^{\oplus 2} \oplus (\mathcal{O}_{|L|}(-r-1))^{\oplus 2} \oplus (\Delta_{r-2} \otimes \mathcal{O}_{|L|}(-2)).$$

Before proving Proposition 6.1, we show some lemmas.

Lemma 6.2. *Let \mathcal{T} be the tangent bundle on M , let $c_i(\mathcal{T})$ be its i -th Chern class, then $c_1(\mathcal{T}).c_i(\mathcal{T}) = 0$ for all i .*

Proof. According to Corollary 4.12 we have $c_1(\mathcal{T}) = [\pi^*\mathcal{O}_{|L|}(-1)^{\otimes (g_L-2)}]$. Denote F to be the fiber class of π . It is enough to show that $c_i(\mathcal{T})|_F = 0$. On the other hand, we can choose a representative of F isomorphic to the Jacobian of some smooth curve. The tangent bundles on Jacobians are trivial with all Chern classes to be zero. Hence the lemma. \square

Since $h^0(\Theta) = 1$, we have only one Θ -divisor D_Θ . Let $M_1 = D_\Theta$. We have exact sequences on M .

$$0 \rightarrow \Theta^{-1} \rightarrow \mathcal{O}_M \rightarrow \mathcal{O}_{M_1} \rightarrow 0. \quad (6.1)$$

$$0 \rightarrow \mathcal{O}_M \rightarrow \Theta \rightarrow \mathcal{O}_{M_1}(\Theta) \rightarrow 0. \quad (6.2)$$

$$0 \rightarrow \Theta^{r-1} \rightarrow \Theta^r \rightarrow \mathcal{O}_{M_1}(\Theta^r) \rightarrow 0, \quad r \geq 2. \quad (6.3)$$

Pushing (6.1) forward, we get three isomorphisms of bundles on $|L|$.

$$0 \rightarrow \pi_* \mathcal{O}_M \rightarrow \pi_* \mathcal{O}_{M_1} \rightarrow 0. \quad (6.4)$$

$$0 \rightarrow R^1 \pi_* \mathcal{O}_M \rightarrow R^1 \pi_* \mathcal{O}_{M_1} \rightarrow 0. \quad (6.5)$$

$$0 \rightarrow R^2 \pi_* \Theta^{-1} \rightarrow R^2 \pi_* \mathcal{O}_M \rightarrow 0. \quad (6.6)$$

The isomorphism in (6.4) is because $\pi_* \Theta^{-1} = R^1 \pi_* \Theta^{-1} = 0$ by Lemma 4.8. The morphism in (6.6) at first is a surjective map because the relative dimension of M_1 over $|L|$ is 1 and hence $R^2 \pi_* \mathcal{O}_{M_1} = 0$; then it is an isomorphism because $R^2 \pi_* \Theta^{-1}$ is a line bundle and $R^2 \pi_* \mathcal{O}_M$ is locally free of rank 1 on the open set of smooth curves in $|L|$. And then the morphism in (6.5) has to be an isomorphism because both (6.4) and (6.6) are.

By pushing forward sequence (6.2), we get three isomorphisms of bundles on $|L|$.

$$0 \rightarrow \pi_* \mathcal{O}_M \rightarrow \pi_* \Theta \rightarrow 0. \quad (6.7)$$

$$0 \rightarrow \pi_* \mathcal{O}_{M_1}(\Theta) \rightarrow R^1 \pi_* \mathcal{O}_M \rightarrow 0. \quad (6.8)$$

$$0 \rightarrow R^1 \pi_* \mathcal{O}_{M_1}(\Theta) \rightarrow R^2 \pi_* \mathcal{O}_M \rightarrow 0. \quad (6.9)$$

We have an isomorphism in (6.7) because they both are line bundles isomorphic to $\mathcal{O}_{|L|}$, (6.8) and (6.9) are because $R^j \pi_* \Theta^i = 0$, for all $j, i > 0$. So we have the following lemma.

Lemma 6.3. *On $|L|$, we have*

- (1) $\pi_* \mathcal{O}_M \simeq \pi_* \Theta \simeq \pi_* \mathcal{O}_{M_1} \simeq \mathcal{O}_{|L|}$;
- (2) $R^2 \pi_* \Theta^{-1} \simeq R^2 \pi_* \mathcal{O}_M \simeq R^1 \pi_* \mathcal{O}_{M_1}(\Theta) \simeq \mathcal{O}_{|L|}(-2)$.
- (3) $R^1 \pi_* \mathcal{O}_M \simeq R^1 \pi_* \mathcal{O}_{M_1} \simeq \pi_* \mathcal{O}_{M_1}(\Theta)$, and they are of rank 2 and Euler characteristic 0.
- (4) $R^1 \pi_* \mathcal{O}_{M_1}(\Theta^i) = 0$, for all $i \geq 2$.

Proof. Statement 1 is trivial.

For statement 2: remember that Θ restricted to a generic fiber is the usual θ -bundle on the Jacobian by Lemma 3.0.1 in [8], and hence we have $(D_\Theta)^g.F = g!$. By Corollary 4.12 we know that $c_1(\mathcal{T}_M) = 0$ since $g_L = 2$. Hence by Hirzebruch-Riemann-Roch, we have $\chi(\Theta) = -\chi(\Theta^{-1})$. On the other

hand we know that $\chi(\Theta) = \sum (-1)^i \chi(R^i \pi_* \Theta) = \chi(\pi_* \Theta) = 1$. So as a result $\chi(\Theta^{-1}) = \chi(R^2 \pi_* \Theta^{-1}) = -1$, so the statement.

For statement 3: from Lemma 6.2 and Hirzebruch-Riemann-Roch we know that $\chi(\mathcal{O}_M) = c_1(\mathcal{T}) \cdot c_2(\mathcal{T}) = 0$, hence $\chi(R^1 \pi_* \mathcal{O}_M) = \chi(\pi_* \mathcal{O}_M) + \chi(R^2 \pi_* \mathcal{O}_M) = 0$.

At last we push (6.3) forward and get $R^1 \pi_* \mathcal{O}_{M_1}(\Theta^r) = 0$, for $r \geq 2$. \square

Push (6.3) forward and we get an exact sequence of bundles on $|L|$.

$$0 \rightarrow \pi_* \Theta^{r-1} \rightarrow \pi_* \Theta^r \rightarrow \pi_* \mathcal{O}_{M_1}(\Theta^r) \rightarrow 0. \quad \text{for } r \geq 2. \quad (6.10)$$

We have already seen that $\pi_* \Theta \simeq \mathcal{O}_{|L|}$. To get the recursion formula, it is enough to compute the splitting type of $\pi_* \mathcal{O}_{M_1}(\Theta^r)$ for all $r \geq 2$.

We define two other determinant line bundles associated to $\mathcal{O}_X(E_2 - E_1)$ and $\mathcal{O}_X(E_1 - E_2)$ on X respectively. Let $\eta_1 = \lambda_{[\mathcal{O}_X(E_2 - E_1)]}$ and $\eta_2 = \lambda_{[\mathcal{O}_X(E_1 - E_2)]}$. According to Lemma 3.0.1 in [8], there is a natural global section of η_1 (resp. η_2) whose vanishing locus consists of all $[\mathcal{F}]$ such that $H^0(\mathcal{F} \otimes \mathcal{O}_X(E_2 - E_1)) \neq 0$ (resp. $H^0(\mathcal{F} \otimes \mathcal{O}_X(E_1 - E_2)) \neq 0$). We denote the two divisors associated to those two natural global sections as D_1 and D_2 respectively.

Remark 6.4. *Since $[\mathcal{O}_X(E_1 - E_2)] + [\mathcal{O}_X(E_2 - E_1)] = 2[\mathcal{O}_X] - 2[\mathcal{O}_{pt}]$, we have $\eta_1 \otimes \eta_2 \simeq \Theta^2(2)$ on M .*

Let $\Pi := D_1 \cap M_1$ and $\Sigma := D_2 \cap M_1$.

Now let \mathcal{C} be the universal curve in $X \times |L|$ and q the projection from $X \times |L|$ to X . Then $\mathcal{O}_{\mathcal{C}} \otimes q^* \mathcal{O}_X(E_1)$ is a flat family of sheaves over $|L|$ and induces a morphism from $|L|$ to M which is a section of π . The image of this morphism, we denote it Π_1 , is contained in $\Pi = D_1 \cap M_1$. And let $\Pi_2 = \overline{\Pi} - \Pi_1$. We define similarly Σ_1 and Σ_2 : Σ_1 is the image of $|L|$ via the morphism induced by the flat family $\mathcal{O}_{\mathcal{C}} \otimes q^* \mathcal{O}_X(E_2)$ on $X \times |L|$, and $\Sigma_2 := \overline{\Sigma} - \Sigma_1$.

Both Π_1 and Σ_1 are isomorphic to $|L| \simeq \mathbb{P}^1$. $\Pi_1 \cap \Sigma_1 = \emptyset$ because E_1 and E_2 intersect every curve in $|L|$ at two different points. For Π_2 and Σ_2 , we have the following lemma.

Lemma 6.5. *Π_2 is also isomorphic to $|L|$ and provides a section of π as well. The same is true for Σ_2 .*

Proof. Because E_1 and E_2 do not intersect each other, they intersect every curve at two different points. And because curves in $|L|$ are of genus 2, any two different points are not linearly equivalent. So for $i = 1, 2$, η_i restricted to a fiber is algebraically but not linearly equivalent to the usual θ -bundle. Moreover according to basic theory of Jacobians, we know that the intersection number of Π with a fiber of π is 2.

So π is a morphism of degree 2 and when restricted on $\overline{\Pi - \Pi_1}$ it is a morphism of degree 1 over \mathbb{P}^1 , hence an isomorphism. So $\Pi_2 = \overline{\Pi - \Pi_1}$ is isomorphic to $|L|$ and provides a section of π . It is analogous for Σ_2 . \square

Let C be any curve in $|L|$. We denote p_C^i the point where E_i meets C . C is smooth at p_C^i . For any point $q_C^1 \in C$, such that $h^0(q_C^1 - p_C^1 + p_C^2) \neq 0$, i.e. $[\mathcal{O}_C(q_C^1)] \in \Pi$, there is another point $q_C^2 \in C$ satisfying that $q_C^1 + p_C^2$ is linearly equivalent to $p_C^1 + q_C^2$ on C . Hence if $p_C^2 \neq q_C^2$, $q_C^1 \neq p_C^1$, then $h^0(q_C^1 + p_C^2) \geq 2$. And hence by Riemann-Roch, we know that $h^1(q_C^1 + p_C^2) = h^0(\omega_C - q_C^1 - p_C^2) \geq 1$, and hence $\omega_C \sim q_C^1 + p_C^2$ since C is of genus 2 and the canonical sheaf ω_C on C is of degree 2. So q_C^1 has either to be p_C^1 or satisfies that $\omega_C \sim p_C^2 + q_C^1$. And if $q_C^1 = p_C^1$, then we have $q_C^2 = p_C^2$ and $\omega_C \sim p_C^1 + p_C^2$. Hence we can assume that $q_C^1 \neq p_C^1$ for a generic C , and hence $\Pi_1 \neq \Pi_2$, $\Sigma_1 \neq \Sigma_2$.

Hence we can specify the universal sheaf on $X \times \Pi_2$ (resp. $X \times \Sigma_2$) as $\mathcal{O}_C \otimes q^* \mathcal{O}_X(K + L - E_2)$ (resp. $\mathcal{O}_C \otimes q^* \mathcal{O}_X(K + L - E_1)$). This is because $\mathcal{O}_C(K + L) \simeq \omega_C$ for all $[C] \in |L|$, and $\omega_C \sim p_C^2 + q_C^1$ which implies that $\mathcal{O}_C(K + L - E_2) \sim \mathcal{O}_C(q_C^1)$.

Lemma 6.6. *For $i = 1, 2$ we have $\pi_*(\Theta^r|_{\Pi_i}) \simeq \mathcal{O}_{|L|}(-r\chi(\mathcal{O}_X)) = \mathcal{O}_{|L|}(-r)$, which is equivalent to saying that $D_{\Theta}\Pi_i = -1$. And the same holds for Σ_i , $i = 1, 2$.*

Proof. By the universal property of Θ we have that $\Theta|_{\Pi_1} = (\det R^\bullet p \mathcal{U}^1)^{-1}$ where $\mathcal{U}^1 \simeq \mathcal{O}_C \otimes q^* \mathcal{O}_X(E_1)$ is the universal sheaf on $X \times \Pi_1$. And also we have the exact sequence on $X \times |L|$:

$$0 \rightarrow p^* \mathcal{O}_{|L|}(-1) \otimes q^* \mathcal{O}_X(-L + E_1) \rightarrow q^* \mathcal{O}_X(E_1) \rightarrow \mathcal{U}_1 \rightarrow 0.$$

So

$$\det R^\bullet p \mathcal{U}_1 \simeq \det R^\bullet p (q^* \mathcal{O}_X(E_1)) \otimes (\det R^\bullet p (p^* \mathcal{O}_{|L|}(-1) \otimes q^* \mathcal{O}_X(-L + E_1)))^{-1}.$$

Then we have

$$\begin{aligned} \det R^\bullet p (q^* \mathcal{O}_X(E_1)) &\simeq \mathcal{O}_{|L|}, \\ \det R^\bullet p (p^* \mathcal{O}_{|L|}(-1) \otimes q^* \mathcal{O}_X(-L + E_1)) &\simeq \mathcal{O}_{|L|}(-1)^{\otimes \chi(\mathcal{O}_X(-L + E_1))}. \end{aligned}$$

$\chi(\mathcal{O}_X(-L + E_1)) = \chi(\mathcal{O}_X(E_1)) - \chi(\mathcal{O}_C(E_1)) = \chi(\mathcal{O}_X(E_1))$, since C is a curve of genus 2 and $\mathcal{O}_C(E_1)$ is a line bundle of degree 1 on C . By Hirzebruch-Riemann-Roch we know that $\chi(\mathcal{O}_X(E_1)) = \chi(\mathcal{O}_X) = 1$.

For Π_2 , we use $\mathcal{O}_C \otimes q^*\mathcal{O}_X(K + L - E_2)$ as the universal sheaf. Similar computation shows that $D_{\Theta}.\Pi_2 = -\chi(\mathcal{O}_X(K + L - E_2)) = -\chi(\mathcal{O}_X)$ since $K.(K + L) = 2g_L - 2 = 2$.

For Σ_i the argument is analogous. \square

$\Pi + \Sigma \sim (2D_{\Theta} + 2F)|_{D_{\Theta}}$. Lemma 6.6 implies that $(\Pi + \Sigma).D_{\Theta} = -4$. Moreover $F.D_{\Theta}^2 = g! = 2$, hence we have $2D_{\Theta}^3 + 4 = (\Pi + \Sigma).D_{\Theta} = -4$. Then we get the following proposition immediately.

Proposition 6.7. *On the moduli space M , we have $D_{\Theta}^3 = -4$.*

Since we know that $\chi(\Theta) = 1$, by Proposition 6.7 we can compute $\chi(\Theta^r(n))$ for all r and n . And we have

$$\chi(\Theta^r(n)) = -\frac{2}{3}r^3 + nr^2 + \frac{5}{3}r. \quad (6.11)$$

However, if we want to write down explicitly the splitting type of $\pi_*\Theta^r$ and get a result which is not only numerical but also gives some geometric description, we have to see how the four projective lines, Π_1 , Π_2 , Σ_1 and Σ_2 intersect each other. It is obvious that $\Pi_1 \cap \Sigma_1 = \emptyset$ because E_1 and E_2 intersect every curve in $|L|$ at two different points. We have several lemmas:

Lemma 6.8. Π_2 has no intersection with Σ_2 , i.e. $\Pi_2.\Sigma_2 = 0$.

Proof. Let C be any curve in $|L|$. As we mentioned before, if $[\mathcal{O}_C(q_C^1)] \in \Pi_2$ and $[\mathcal{O}_C(q_C^2)] \in \Sigma_2$, then $q_C^1 + p_C^2 \sim p_C^1 + q_C^2$ with p_C^i the point where C meets E_i . Since $p_C^1 \neq p_C^2$, and $p_C^1 - p_C^2 \sim q_C^1 - q_C^2$, we have $q_C^1 \neq q_C^2$ for any $[C] \in |L|$ and hence the lemma. \square

Now we compute $\Pi_1.\Sigma$ and $\Pi.\Sigma_1$.

Notice that the universal sheaf \mathcal{U}^1 over $X \times \Pi_1$ can be chosen to be $\mathcal{O}_C \otimes q^*\mathcal{O}_X(E_1)$, as a result $[\mathcal{F}] \in \Pi_1 \cap \Sigma \Leftrightarrow H^0(\mathcal{O}_{C_{\mathcal{F}}} \otimes q^*\mathcal{O}_X(E_1) \otimes q^*\mathcal{O}_X(E_1 - E_2)) \neq 0$, where $C_{\mathcal{F}}$ is the supporting curve of \mathcal{F} . It is analogous for $\Pi \cap \Sigma_1$.

Let $\mathcal{B}^1 = \mathcal{O}_C \otimes q^*\mathcal{O}_X(2E_1 - E_2)$, $\mathcal{B}^2 = \mathcal{O}_C \otimes q^*\mathcal{O}_X(2E_2 - E_1)$. These two sheaves are also flat families over $X \times |L|$ hence induce two embeddings mapping $|L|$ to M which both are sections of π . Denote their image in M as P_1 and P_2 respectively. $P_i \simeq \mathbb{P}^1$.

Lemma 6.9. $\Theta|_{P_i} \simeq \mathcal{O}_{\mathbb{P}^1}(-\chi(\mathcal{O}_X) + 2) = \mathcal{O}_{\mathbb{P}^1}(1)$, for $i = 1, 2$.

Proof. The proof is analogous to Lemma 6.6, and instead of $\chi(-L+E_1)$ we have $\chi(-L+2E_1-E_2)$ or $\chi(-L+2E_2-E_1)$ which are equal to $\chi(-L+E_1)-2$. \square

Lemma 6.10. For any curve \mathbf{C} in M , let $d = \deg \Theta|_{\mathbf{C}}$,

(1) If $d < 0$, then $\mathbf{C} \subset M_1$.

(2) If $d \geq 0$, and also \mathbf{C} is not contained in M_1 , then $d = \#(\mathbf{C} \cap M_1)$, counting with multiplicity.

Proof. If the curve is not contained in $M_1 = D_\Theta$, then there is a nonzero global section of Θ vanishing at points corresponding to sheaves with global sections. Hence the degree of Θ restricted to that curve should be nonnegative and must equal to $\mathbf{C} \cap M_1$ counting with multiplicity. \square

Remark 6.11. Because of Lemma 6.10, if P_1 (resp. P_2) is not contained in M_1 , then $\Pi_1.\Sigma = \#\Pi_1 \cap \Sigma = 1$ (resp. $\Pi.\Sigma_1 = \#\Pi \cap \Sigma_1 = 1$).

Lemma 6.12. Neither P_1 nor P_2 is contained in M_1 .

Proof. Note that a priori, P_i is contained in D_i for $i = 1, 2$. If P_1 is contained in M_1 , then $P_1 \subset M_1 \cap D_1 = \Pi$. Hence P_1 has to be either Π_1 or Π_2 . But Θ restricted on P_1 has degree 1 while restricted on Π_i it has degree -1 by Lemma 6.6. So we know that P_1 can not be contained in M_1 . For P_2 it is analogous. \square

Because of Lemma 6.12 and Remark 6.11, we have $\Pi_1.\Sigma = \Pi.\Sigma_1 = 1$. On the other hand, we have $\Pi_1 \cap \Sigma_1 = \emptyset$, $\Pi_2 \cap \Sigma_2 = \emptyset$. Hence we have $\Pi_1.\Sigma_2 = 1$ and $\Pi_2.\Sigma_1 = 1$. We now only need to compute $\Pi_1.\Pi_2$ and $\Sigma_1.\Sigma_2$.

Recall that $D_\Theta = M_1$. Now on M_1 we have an exact sequence.

$$0 \rightarrow \eta_1^{-1} \otimes \eta_2^{-1} \rightarrow \mathcal{O}_{M_1} \rightarrow \mathcal{O}_{M_2} \rightarrow 0. \quad (6.12)$$

M_2 is a subscheme of M_1 , which equals to $\Pi + \Sigma$ as a divisor. $\Pi + \Sigma \sim (2D_\Theta + 2F)|_{D_\Theta}$. Because of Remark 6.4 we can rewrite sequence (6.12) as follows:

$$0 \rightarrow \Theta^{-2}(-2)|_{M_1} \rightarrow \mathcal{O}_{M_1} \rightarrow \mathcal{O}_{M_2} \rightarrow 0. \quad (6.13)$$

Using formula (6.11), by a direct computation we get $\chi(\mathcal{O}_{M_2}) = 2$. Hence the arithmetic genus of M_2 is negative. Also we know that $M_2 = \Pi_1 + \Pi_2 + \Sigma_1 + \Sigma_2$, and the Π_i and the Σ_i are isomorphic to \mathbb{P}^1 . So M_2 can not be connected and therefore $\Pi_1 \cap \Pi_2 = \Sigma_1 \cap \Sigma_2 = \emptyset$.

Remark 6.13. *So the picture of these four curves is very clear: $\Pi_1 \cap \Pi_2 = \emptyset = \Sigma_1 \cap \Sigma_2$; $\Pi_1 \cdot \Sigma_2 = 1$ and $\Pi_2 \cdot \Sigma_1 = 1$; and $\Pi_1 \cap \Sigma_1 = \Pi_2 \cap \Sigma_2 = \emptyset$.*

We have the exact sequence on M_2 as follows.

$$0 \rightarrow (\mathcal{O}_{\Pi_1}(-1) \oplus \mathcal{O}_{\Pi_2}(-1)) \otimes \Theta^r \rightarrow \mathcal{O}_{M_2}(\Theta^r) \rightarrow (\mathcal{O}_{\Sigma_1} \oplus \mathcal{O}_{\Sigma_2}) \otimes \Theta^r \rightarrow 0 \quad (6.14)$$

We then have the following proposition.

Proposition 6.14. $\pi_* \mathcal{O}_{M_2}(\Theta^r) \simeq \mathcal{O}_{|L|}(-1-r)^{\oplus 2} \oplus \mathcal{O}_{|L|}(-r)^{\oplus 2}$.

Proof. By Lemma 6.6 we have $\pi_*(\Theta^r|_{\Pi_i}) \simeq \pi_*(\Theta^r|_{\Sigma_i}) \simeq \mathcal{O}_{|L|}(-r)$, for $i = 1, 2$. So push (6.14) forward and we get

$$0 \rightarrow \mathcal{O}_{|L|}(-1-r)^{\oplus 2} \rightarrow \pi_* \mathcal{O}_{M_2}(\Theta^r) \rightarrow \mathcal{O}_{|L|}(-r)^{\oplus 2} \rightarrow 0 \quad (6.15)$$

It is easy to see there are no higher direct image along π for sheaves on M_2 , since π restricted on M_2 has relative dimension zero. And sequence (6.15) splits for every r . \square

We tensor the sequence (6.13) by some power of Θ . Then we have following exact sequences on M_1 .

$$0 \rightarrow \mathcal{O}_{M_1}(\Theta^{-2}(-2)) \rightarrow \mathcal{O}_{M_1} \rightarrow \mathcal{O}_{M_2} \rightarrow 0. \quad (6.16)$$

$$0 \rightarrow \mathcal{O}_{M_1}(\Theta^{-1}(-2)) \rightarrow \mathcal{O}_{M_1}(\Theta) \rightarrow \mathcal{O}_{M_2}(\Theta) \rightarrow 0. \quad (6.17)$$

$$0 \rightarrow \mathcal{O}_{M_1}(\Theta^{r-2}(-2)) \rightarrow \mathcal{O}_{M_1}(\Theta^r) \rightarrow \mathcal{O}_{M_2}(\Theta^r) \rightarrow 0, \quad r \geq 0. \quad (6.18)$$

Push all of them forward and we get

$$0 \rightarrow \pi_* \mathcal{O}_{M_1} \rightarrow \pi_* \mathcal{O}_{M_2} \rightarrow R^1 \pi_* \mathcal{O}_{M_1}(\Theta^{-2}) \otimes \mathcal{O}_{|L|}(-2) \rightarrow R^1 \pi_* \mathcal{O}_{M_1} \rightarrow 0. \quad (6.19)$$

$$0 \rightarrow \pi_* \mathcal{O}_{M_1}(\Theta) \rightarrow \pi_* \mathcal{O}_{M_2}(\Theta) \rightarrow R^1 \pi_* \mathcal{O}_{M_1}(\Theta^{-1}) \otimes \mathcal{O}_{|L|}(-2) \rightarrow R^1 \pi_* \mathcal{O}_{M_1}(\Theta) \rightarrow 0. \quad (6.20)$$

$$0 \rightarrow \pi_* \mathcal{O}_{M_1}(\Theta^{r-2}) \otimes \mathcal{O}_{|L|}(-2) \rightarrow \pi_* \mathcal{O}_{M_1}(\Theta^r) \rightarrow \pi_* \mathcal{O}_{M_2}(\Theta^r) \rightarrow R^1 \pi_* \mathcal{O}_{M_1}(\Theta^{r-2}) \otimes \mathcal{O}_{|L|}(-2) \rightarrow 0, r \geq 2. \quad (6.21)$$

In (6.19) and (6.20), the zeros on the right are because $R^1 \pi_* \mathcal{O}_{M_2}(\Theta^r) = 0$ for all r . The left zeros are because $\pi_* \mathcal{O}_{M_1}(\Theta^{-r}) = 0$, $\forall r \geq 1$. In (6.21) the right zero is because $R^1 \pi_* \mathcal{O}_{M_1}(\Theta^r) = 0$ as $r \geq 2$ by Lemma 6.3. And (6.21) will be a short exact sequence with three terms when $r \geq 4$. Then we have a simple corollary of Proposition 6.14.

Corollary 6.15. *The canonical sheaf ω_M on M is trivial.*

Proof. Since by Corollary 4.12 we already know that $c_1(\mathcal{T}_M) = 0$, it is enough to show $h^0(\omega_M) = h^3(\mathcal{O}_M) = 1$.

From Proposition 6.14 and Statement 3 in Lemma 6.3 and also sequence (6.20), we can see that $\chi(\pi_*\mathcal{O}_{M_1}(\Theta)) = 0$, and there is a injective morphism from $\pi_*\mathcal{O}_{M_1}(\Theta)$ to $\pi_*\mathcal{O}_{M_2}(\Theta) \simeq \mathcal{O}_{|L|}(-1)^{\oplus 2} \oplus \mathcal{O}_{|L|}(-2)^{\oplus 2}$. Hence $\pi_*\mathcal{O}_{M_1}(\Theta) \simeq \mathcal{O}_{|L|}(-1)^{\oplus 2}$. Also according to Lemma 6.3, we have $\pi_*\mathcal{O}_{M_1}(\Theta) \simeq R^1\pi_*\mathcal{O}_{M_1} \simeq R^1\pi_*\mathcal{O}_M \simeq \mathcal{O}_{|L|}(-1)^{\oplus 2}$, and $\pi_*\mathcal{O}_{M_1} \simeq \mathcal{O}_{|L|}$. Hence $H^1(R^1\pi_*\mathcal{O}_{M_1}) = H^2(\pi_*\mathcal{O}_{M_1}) = 0$. On the other hand, since π restricted on M_1 is of relative dimension 1, we have $R^i\pi_*\mathcal{O}_{M_1} = 0$ for all $i \geq 2$. Hence by the spectral sequence we know that $H^2(\mathcal{O}_{M_1}) = 0$.

From sequence (6.1) we have the exact sequence as follows

$$H^2(\mathcal{O}_{M_1}) \rightarrow H^3(\Theta^{-1}) \rightarrow H^3(\mathcal{O}_M) \rightarrow 0.$$

Because $R^2\pi_*\Theta^{-1} \simeq \mathcal{O}_{|L|}(-2)$ and $R^i\pi_*\Theta^{-1} = 0$ for all $i < 2$, we have $h^3(\Theta^{-1}) = h^1(R^2\pi_*\Theta^{-1}) = 1$; together with the vanishing of $H^2(\mathcal{O}_{M_1})$, we get $h^3(\mathcal{O}_M) = h^3(\Theta^{-1}) = 1$. \square

Corollary 6.15 gives us an interesting result in the theory of compactified Jacobians of integral curves with planar singularities as follows.

Corollary 6.16. *Let X be any simply connected smooth projective surface over \mathbb{C} , L be an effective line bundle satisfying (\mathbf{A}'_1) and (\mathbf{A}'_2) , moreover $\dim |L| \geq 3$ and $g_L = 2$, then for a generic integral curve C in $|L|$, the compactified Jacobian J^{g_L-1} which parametrizes the rank one torsion free sheaves of Euler characteristic zero has its dualizing sheaf be the trivial line bundle.*

Proof of Proposition 6.1. As stated in the proof of Corollary 6.15, we already know that $\pi_*\mathcal{O}_{M_1}(\Theta) \simeq R^1\pi_*\mathcal{O}_{M_1} \simeq \mathcal{O}_{|L|}(-1)^{\oplus 2}$. We rewrite (6.21) with $r = 2$ as

$$0 \rightarrow \mathcal{O}_{|L|}(-2) \rightarrow \pi_*\mathcal{O}_{M_1}(\Theta^2) \rightarrow \mathcal{O}_{|L|}(-3)^{\oplus 2} \oplus \mathcal{O}_{|L|}(-2)^{\oplus 2} \rightarrow \mathcal{O}_{|L|}(-3)^{\oplus 2} \rightarrow 0. \quad (6.22)$$

Hence $\pi_*\mathcal{O}_{M_1}(\Theta^2) \simeq \mathcal{O}_{|L|}(-2)^{\oplus 3}$, together with sequence (6.10) we get the expression for $\pi_*\Theta^2$. Lemma 6.3 also says that $R^1\pi_*\mathcal{O}_{M_1}(\Theta) \simeq \mathcal{O}_{|L|}(-2)$. So sequence (6.21) with $r = 3$ implies that $\pi_*\mathcal{O}_{M_1}(\Theta^3) \simeq \mathcal{O}_{|L|}(-4) \oplus \mathcal{O}_{|L|}(-3)^{\oplus 4}$. Then we know the splitting type of $\pi_*\Theta^3$.

For Θ^r , $r \geq 4$, both (6.10) and (6.21) are short exact sequences with three terms and split, which implies Statements 1 and 3 in the proposition. \square

We have defined $Z^r(t) = \sum_n h^0(M, \lambda_{c_n^r})t^n = \sum_n h^0(M, \Theta^r \otimes \pi^* \mathcal{O}_{|L|}(n))t^n$. The generating function $Z^r(t)$ can be written down explicitly as follows:

1. $Z^1(t) = \frac{1}{(1-t)^2}$; $Z^2(t) = \frac{1+3t^2}{(1-t)^2}$; $Z^3(t) = \frac{1+3t^2+4t^3+t^4}{(1-t)^2}$.
2. for $r \geq 4$, $Z^r(t) = Z^{r-1}(t) + (Z^{r-2}(t) - Z^{r-3}(t)) \cdot t^2 + \frac{2t^r+2t^{r+1}}{(1-t)^2}$.

The recursion formula 2 implies that

$$Z^r(t) = \frac{1 + 3t^2 + \sum_{i=3}^r ((i+1)t^i + (i-2)t^{i+1})}{(1-t)^2} \text{ for } r \geq 2.$$

Remark 6.17. *These results are compatible with Statement 2 in Theorem 4.5.2 in [8] as $X = \mathbb{P}(\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(-e))$ and $L = 2G + (e+3)F$ with $e = 0, 1$.*

Proof of Theorem 1.3. In this case we have

$$Y_{g_L=2}^r(t) = \frac{Z^r(t)}{(1-t)^{l-1}},$$

and hence the theorem. □

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